Note on Charge and Field Fluctuations

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The field fluctuations which arise from the possibility of creating positron-electron pairs are computed. These fluctuations, which are inescapable in all field measurements, give a simple interpretation of Heisenberg's results on charge fluctuations, and of the divergences which appear in his calculations. The pair-induced fluctuations in the radiation field are in general small of order α compared to those which arise from the corpuscular character of the radiation.

THE fluctuations in charge density to be expected on the basis of the positron theory of Dirac have recently been studied by Heisenberg.[1] Consider a large enclosure of volume V, in which there are known to be N_k electrons of momentum $\mathbf{k}$, and M_k positrons of momentum $\mathbf{k}$. Consider further a small volume of dimensions L_1, L_2, L_3, and ask for the average value of the charge in this volume over a time T. This is, of course, with e the electronic charge,

$$\bar{e}=e(\bar{m}-\bar{n});\quad \bar{n}=(L_1L_2L_3/V)\sum_k N_k;\quad \bar{m}=(L_1L_2L_3/V)\sum_k M_k.$$

The charge in this volume will, however, fluctuate about its mean value, and these fluctuations $(\Delta e)^2$ can now be divided, as Heisenberg has shown, into three sets of terms:

(α) vacuum terms, which are present when $N=M=0$, and which have no analogue in non-relativistic theory.

(β) terms linear in N_k, M_k. These reduce to their classically expected value

$$(\Delta e)^2=e^2(\bar{n}+\bar{m}) \tag{1}$$

for space time regions large compared to those within which the electrons and positrons can be localized. When, however, for all electrons and positrons present

$$T\ll\hbar(m^2c^4+c^2k^2)^{-\frac{1}{2}};\quad L_i\ll\hbar(m^2c^2+k_i^2)^{-\frac{1}{2}}$$

(k_i the component of momentum parallel to L_i), then these fluctuations tend to vanish. The deviations from (1) under these circumstances are an immediate consequence of the limitations on the localizability of the particles.

(γ) negative terms quadratic in the N's and M's, which give the familiar reduction, to be expected on the basis of the exclusion principle, in the fluctuations of a dense Fermi gas.

It is with the vacuum terms that Heisenberg is chiefly concerned. These not only do not vanish, but for a sharply limited spatio temporal region are infinite. This result can only be interpreted as a consequence of the infinite disturbance produced when we try to measure the charge in a sharply defined region; and Heisenberg has, in fact, shown that if we take the boundaries of our spatial volume "spread out" over a distance b, then we get in general finite fluctuations. Thus

$$\begin{aligned}(\Delta e)^2&\sim e^2L^2/bcT \text{ for } cT\ll b\ll L;\ cT\ll\lambda=\hbar/mc.\\ &\sim e^2L^2\lambda/bc^2T^2 \text{ for } cT\ll b\ll L;\ cT\gg\lambda.\end{aligned} \tag{2}$$

These conclusions lose at once their paradoxical character, if we consider the experimental arrangement by which we might measure the charge. To do this it is only necessary to measure the normal component of the electric field over the surface of the volume in which we are interested. These field measurements will give fluctuating results, and these fluctuations will correspond to those of the charge within the volume. We have thus to consider the fluctuations in the longitudinal component of the electric intensity, averaged over a time T, and a slab of dimensions L_1, L_2, L_3.

These field fluctuations are given by an integral of the form

$$(\Delta\mathcal{E}_{||})^2=\frac{2^6\pi^{-4}e^2}{L_1^2L_2^2L_3^2c^2T^2}\int d\mathbf{k}\int d\mathbf{k}'\frac{\epsilon\epsilon'-\mathbf{k}\cdot\mathbf{k}'-\mu^2}{\epsilon\epsilon'|\mathbf{k}-\mathbf{k}'|^2}\Pi \tag{3}$$

[1] W. Heisenberg, Verh. d. Sächs. Akad. **86**, 317 (1934).

with

$$\epsilon=(\mu^2+k^2)^{\frac{1}{2}};\quad \epsilon'=(\mu^2+k'^2)^{\frac{1}{2}}\quad\text{and}\quad \mu=mc/\hbar=1/\lambda.$$

$$\Pi=\frac{\sin^2(c/2)(\epsilon+\epsilon')T}{(\epsilon+\epsilon')^2}\cdot\frac{\sin^2\frac{1}{2}(k_1-k_1')L_1}{(k_1-k_1')^2}\cdot\frac{\sin^2\frac{1}{2}(k_2-k_2')L_2}{(k_2-k_2')^2}\cdot\frac{\sin^2\frac{1}{2}(k_3-k_3')L_3}{(k_3-k_3')^2}.$$

The integrand differs from that given by Heisenberg for the charge fluctuations essentially by the factor $|\mathbf{k}-\mathbf{k}'|^{-2}$. This integral converges, and may be evaluated when we make limiting assumptions about the relative magnitude of the lengths cT, L_i, λ. We thus find

for		$(\Delta\mathcal{E}_{\|\|})^2\sim$
all $L_i\gg\lambda$ $cT\gg\lambda$ $L\lesseqgtr cT$		$\dfrac{e^2\lambda}{L_1L_2L_3c^2T^2}$
all $L_i\gg cT$ $cT\ll\lambda$ $L_i\lesseqgtr\lambda$		$\dfrac{e^2}{L_1L_2L_3cT}$
all $L_i\ll\lambda$ $L_i\ll cT$ $cT\lesseqgtr\lambda$	$L_1\backsimeq L_2\backsimeq L_3\backsimeq L$	$\dfrac{e^2}{L^2c^2T^2}$
all $L_i\ll\lambda$ $L_i\ll cT$ $cT\lesseqgtr\lambda$	$L_1\ll L_2$; $L_2\backsimeq L_3=L$	$\dfrac{e^2}{L^2c^2T^2}\ln(L/L_1)$

(4)

If we apply these results to field measurements in a rectangular slab of area L^2, and thickness b, and use $\sqrt{(\Delta e)^2}\sim L^2\sqrt{(\Delta\mathcal{E}_{\|\|})^2}$, we get at once for the charge fluctuations the values (2) given by Heisenberg, and the further value, for the case $L\ll cT$

$$(\Delta e)^2\sim e^2(L^2/c^2T^2)\ln(L/b),$$

which is also in agreement with Heisenberg's formulae. We thus see that the infinite fluctuations which we find for a sharply limited volume arise from the measurement of fields in infinitely thin slabs; for these an infinite charge density is required, and the charge fluctuations correspond to the pairs created in the neighborhood of these surface charges.

The calculations of field fluctuations can be extended to include those of the radiation field, which are of the same order of magnitude as those of the longitudinal field. Thus the fluctuations in the average value of the total electric intensity are given by an integral of the form (3), but with an extra factor

$$\frac{(\mathbf{k}-\mathbf{k}')^2\{(\mathbf{k}-\mathbf{k}')^2(\epsilon\epsilon'-\mathbf{k}\cdot\mathbf{k}'-\mu^2)+(\epsilon+\epsilon')^2(\epsilon\epsilon'+3\mathbf{k}\cdot\mathbf{k}'+5\mu^2)\}}{(\epsilon\epsilon'-\mathbf{k}\cdot\mathbf{k}'-\mu^2)[(\epsilon+\epsilon')^2-(\mathbf{k}-\mathbf{k}')^2]^2}.$$

The corresponding integrals can again be evaluated under limiting assumptions about the magnitude of L, cT, λ and lead to results whose order of magnitude agrees in every case with (4).

These fluctuations in the radiation field are not in principle separable from the fluctuations arising from the corpuscular character of electromagnetic radiation. These latter fluctuations have been studied by Bohr and Rosenfeld,[2] who give for them

$$(\Delta\epsilon_r)^2\sim\hbar cL^{-4}\qquad\text{for } L\gg cT,$$

$$(\Delta\epsilon_r)^2\sim\hbar cL^{-2}(cT)^{-2}\qquad\text{for } cT\gg L.$$

To these must be added the fluctuations arising from the creation of matter by the field measurements, so that the total fluctuations are, for instance,

$$(\Delta\epsilon)^2\sim\hbar cL^{-2}(cT)^{-2}(1+e^2/\hbar c);\qquad L\ll cT;\ L\ll\lambda.$$

$$(\Delta\epsilon)^2\sim\hbar cL^{-2}(cT)^{-2}(1+e^2/mc^2L);\quad L\ll cT;\quad L\gg\lambda.$$

From the form of these expressions it seems quite doubtful whether the physical significance of the modifications introduced by the pairs can be legitimately evaluated without taking into account the atomicity of the electric charge with which the field measurements are necessarily made.

[2] N. Bohr and L. Rosenfeld, Det. Kgl. Dansk. Vid. Selskap. **12**, 8 (1933).